General Relativity II

Non-euclidean Geometry

- Lobachevsky discovered the geometry of space with negative curvature:

  5. Given a line and a point not on the line, arbitrary many lines can be drawn through that point that will be parallel to the first line.

- Gauss discovered the geometry of space with positive curvature:

  5. Given a line and a point not on the line, no lines can be drawn through that point that will be parallel to the first line.

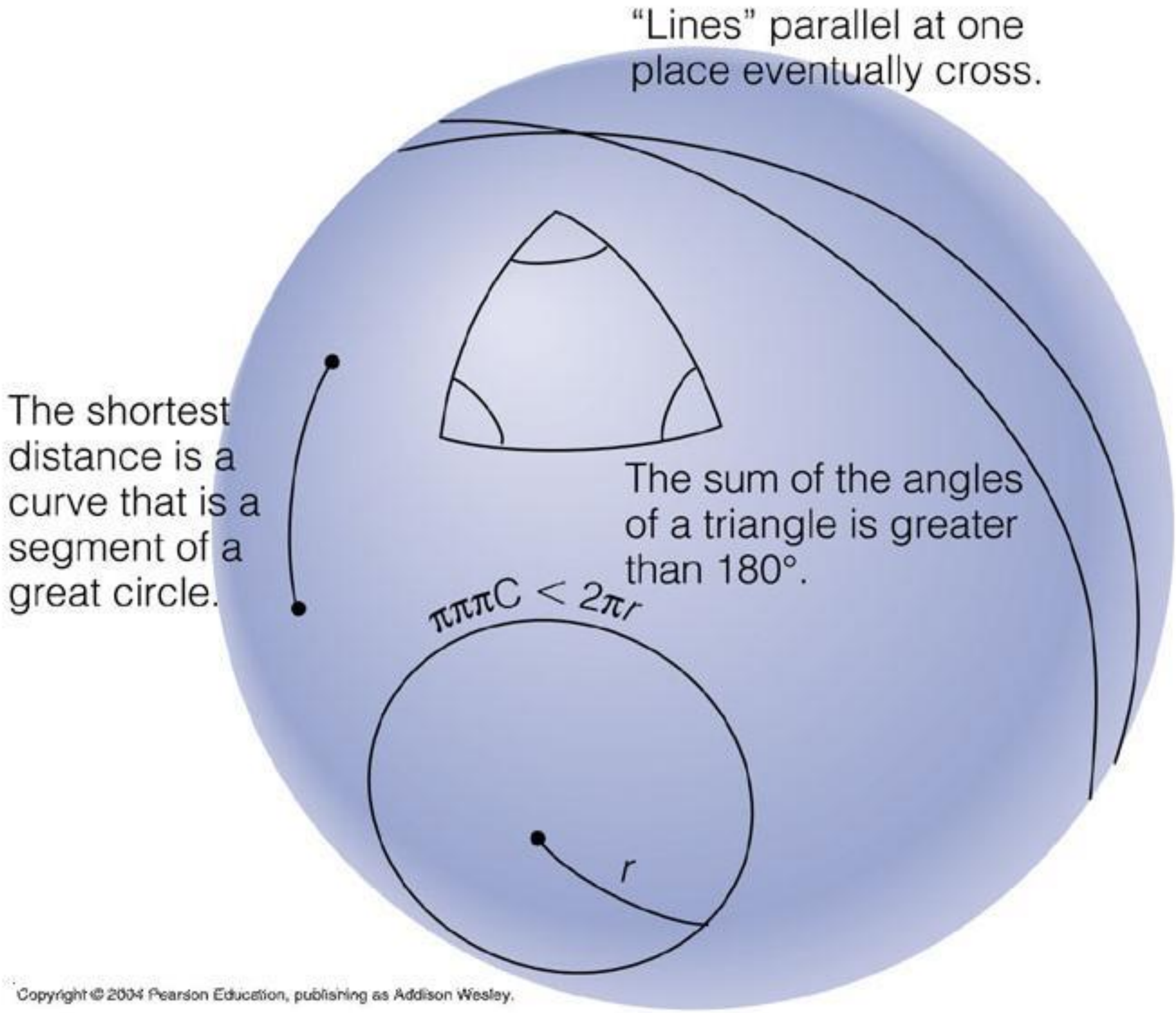
Non-euclidean Geometry

- Sphere has a two-dimensional geometry with positive curvature. Such geometry is sometimes called spherical.
- The geometry of space with negative curvature is sometimes called hyperbolic geometry.
- Spherical geometry is finite.
- Flat geometry is infinite.
- Hyperbolic geometry is even “more infinite” than the flat one.

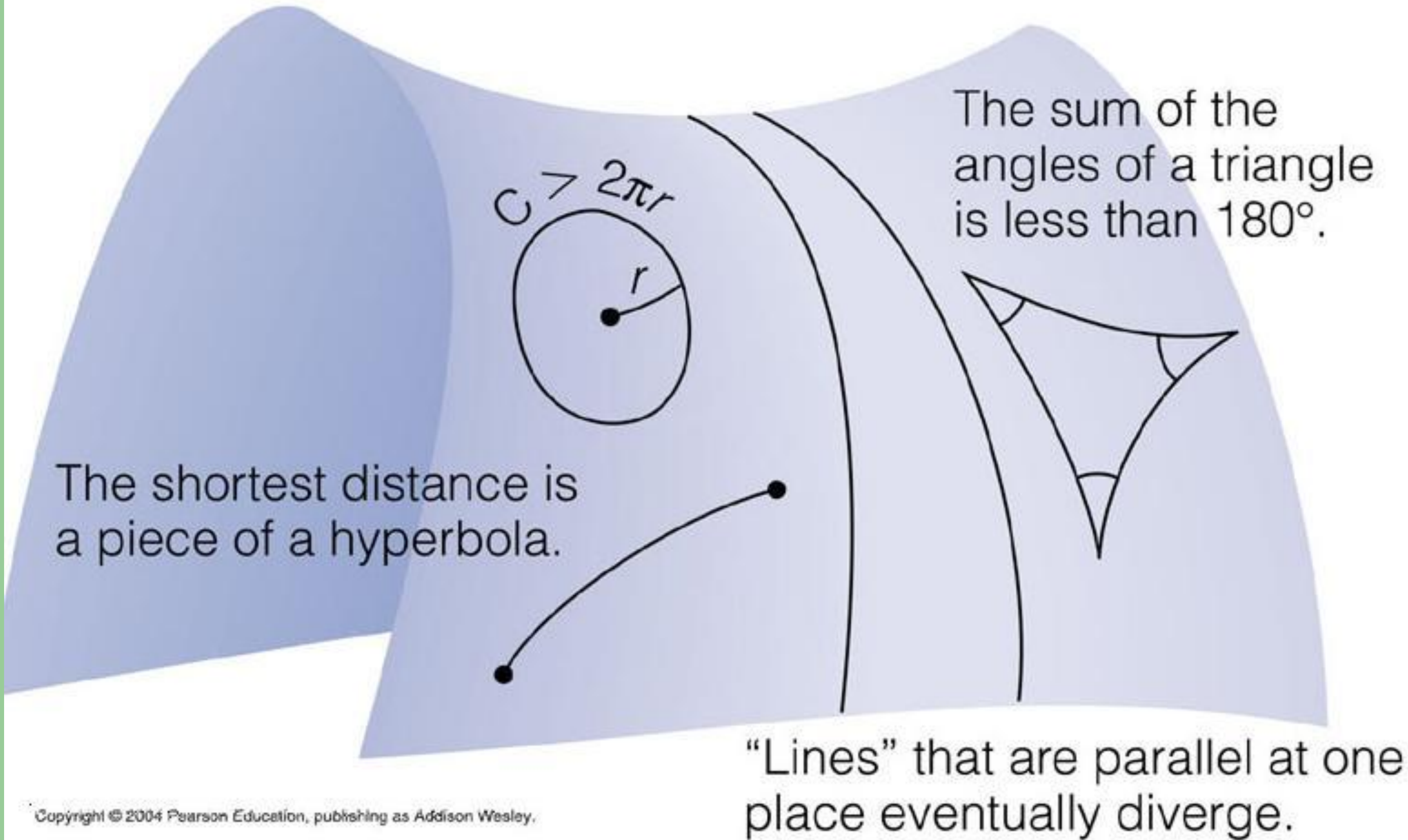
Non-euclidean Geometry

- Sum of angles of a triangle is
  - greater than 180^{o} in spherical geometry.
  - equal to 180^{o} in flat geometry.
  - less than 180^{o} in hyperbolic geometry.
- Circumference of a sphere with radius r is
  - less than $2\pi r$ in spherical geometry.
  - equal to $2\pi r$ in flat geometry.
  - greater than $2\pi r$ in hyperbolic geometry.

Spherical Geometry



Hyperbolic Geometry



Geometry of Space-Time

Now we know, that

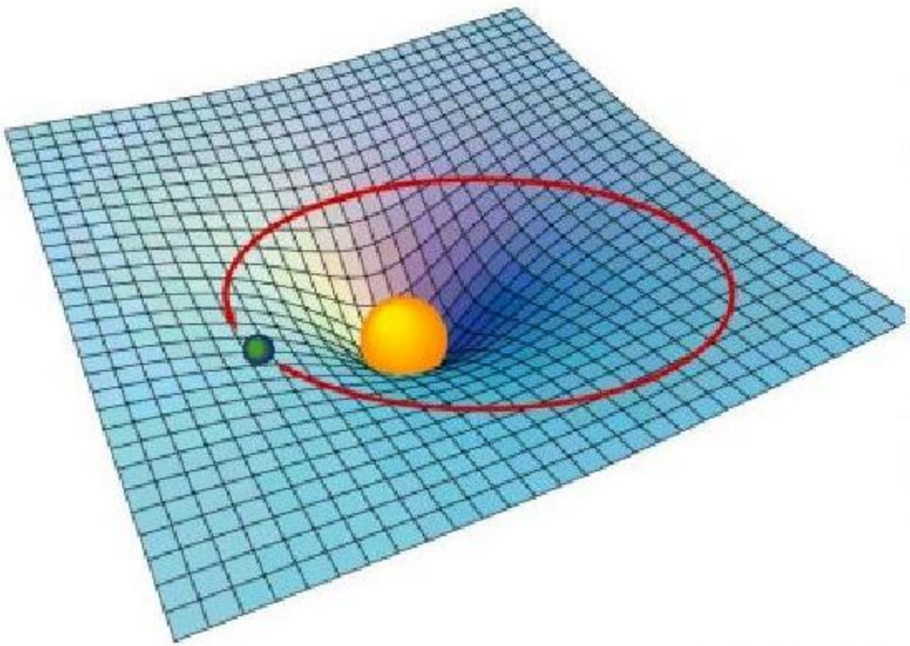
- in flat space, the shortest distance between two points is a straight line.
- in curved space, the shortest distance between two points is not necessarily a straight line.
- in flat (i.e. Minkowski) space-time inertial observers move between two events along straight lines in such a way as to maximize their proper time.

Geometry of Space-Time

- Thus, we can conclude, that in *curved space-time* inertial observers move between two events not necessarily along straight lines, but in such a way as to maximize their proper time.

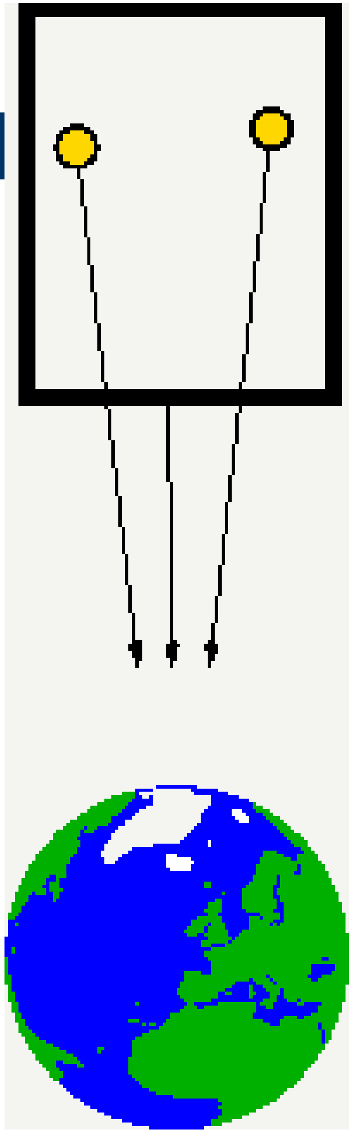
- This type of world line is called a **geodesic**.

Geometry of Space-Time



- Imagine space as a stretched rubber sheet.
- A mass on the surface will cause a deformation.
- Another mass dropped onto the sheet will roll toward that mass.

Tidal Forces



- The scale of a given inertial frame is limited by tidal forces. Since the gravity forces changes with distance, a force acting on one part of an object may be different from the force acting on another part of the same object.
- The tidal forces are responsible for the tides on Earth, because the Moon gravity is different at two sides of the Earth.

Tidal Forces

- The existence of tidal forces means that in the gravitational field, different places within one inertial frame feel slightly different gravity forces, whereas in an accelerated frame all places move with precisely the same acceleration.

Tidal Forces

- Thus, the equivalence principle is not exact – it is only approximate for any object of finite size. It becomes exact only for a point.
- It is said that inertial frames in GR are *local*, i.e. they only extend for a very small distance around an observer.

Einstein Equations: Physics

- Equations of GR relate the curvature of space-time (its ***geometry***), and its contents (***matter + energy***).
- Symbolically, they are

Geometry = Matter + Energy

- Which side is the cause?
  - **A**: geometry
  - **B**: matter + energy

Einstein Equations: Math

$$\frac{1}{2}g^{rs}\left(-\frac{\partial^2 g_{ij}}{\partial x^r \partial x^s}+\frac{\partial^2 g_{is}}{\partial x^r \partial x^j}+\frac{\partial^2 g_{rj}}{\partial x^i \partial x^s}-\frac{\partial^2 g_{rs}}{\partial x^i \partial x^j}\right)+\frac{1}{4}g^{qp}\left(-\frac{\partial g_{is}}{\partial x^p}+\frac{\partial g_{pi}}{\partial x^s}+\right.$$

$$\left.\frac{\partial g_{ps}}{\partial x^i}\right)\times\left(\frac{\partial g_{qj}}{\partial x^r}+\frac{\partial g_{qr}}{\partial x^j}-\frac{\partial g_{rj}}{\partial x^q}\right)-\frac{1}{4}g^{qp}\left(-\frac{\partial g_{ij}}{\partial x^p}+\frac{\partial g_{pi}}{\partial x^j}+\frac{\partial g_{pj}}{\partial x^i}\right)\left(\frac{\partial g_{qr}}{\partial x^s}+\right.$$

$$\left.\frac{\partial g_{qs}}{\partial x^r}-\frac{\partial g_{rs}}{\partial x^q}\right)-\frac{1}{4}g_{ij}g^{rs}g^{uv}\left(-\frac{\partial^2 g_{rs}}{\partial x^u \partial x^v}+\frac{\partial^2 g_{rv}}{\partial x^u \partial x^s}+\frac{\partial^2 g_{us}}{\partial x^r \partial x^v}-\frac{\partial^2 g_{uv}}{\partial x^r \partial x^s}\right)+$$

$$\frac{1}{8}g_{ij}g^{rs}g^{uv}g^{qp}\left(\frac{\partial g_{qr}}{\partial x^v}+\frac{\partial g_{qv}}{\partial x^r}-\frac{\partial g_{rv}}{\partial x^q}\right)\left(\frac{\partial g_{ps}}{\partial x^u}+\frac{\partial g_{pu}}{\partial x^s}-\frac{\partial g_{us}}{\partial x^p}\right)-$$

$$\frac{1}{8}g_{ij}g^{rs}g^{uv}g^{qp}\left(\frac{\partial g_{qr}}{\partial x^s}+\frac{\partial g_{qs}}{\partial x^r}-\frac{\partial g_{rs}}{\partial x^q}\right)\left(\frac{\partial g_{pu}}{\partial x^v}+\frac{\partial g_{pv}}{\partial x^u}-\frac{\partial g_{uv}}{\partial x^p}\right)=\frac{8\pi G}{c^4}T_{ij}.$$

Dynamical Space

- In GR space becomes a dynamical quantity. Space can be curved, perturbed, deformed in arbitrary way, and these deformations can change with time.
- Distortions of space can move – those are called ***gravitational waves***.
- Space can reconnect with itself – ***wormholes***.
- Space can flow into a point of infinite density – ***singularity*** – making a ***black hole***.

Testing GR

- Any physical theory must be constantly tested, and GR is no exception.
- All tests of GR can be separated into two types: a *weak field limit*, i.e. testing GR when deviations from Newton’s gravity are weak, and a *strong field limit*, when deviations from the Newton’s law are large.
- Weak field limit test are numerous (but they are less valuable, because there are alternative theories of gravity).

Weak Field Tests

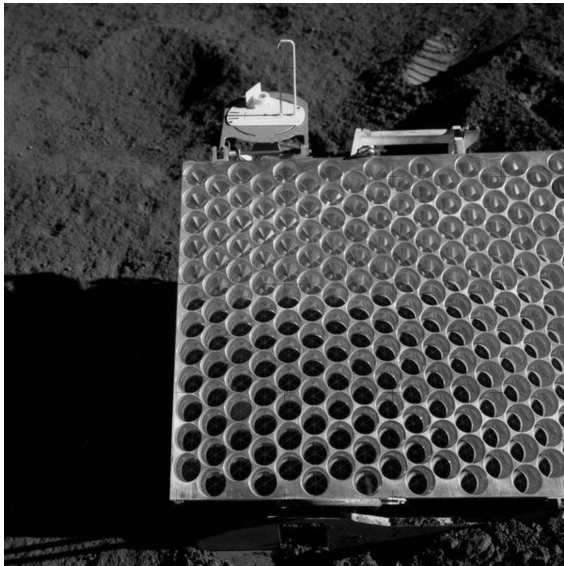
- Precession of planetary orbits.
  - Anomalous precession of Mercury explained by Einstein in 1916.
- Bending of light by the Sun’s gravity field.
  - Measured by Eddington in 1919.

Weak Field Tests

- Time delay due to the Earth gravity.
  - Need to account in time-keeping arose in ~1960.
  - Delay due to Saturn measured by *Cassini* probe to 0.002% in 2003.

Weak Field Tests

- Gravitational redshift.
  - Pound–Rebka experiment at Harvard in 1959.
  - Routine correction for modern GPS systems.
- “Frame dragging”.
  - Gravity Probe B (2005) – to 15%
  - LARES satellite (2012) – to 1%.
- Lunar Laser Ranging Experiment
  - Measures everything.



Strong Field Tests

- Strong field limit tests are much more difficult to perform, but only they can convincingly confirm (or reject!) GR:
  - Gravitational radiation from a binary pulsar.
    - First discovered in 1974 by Joseph Taylor and Russel Hulse (Nobel prize in 1993).
  - Existence of black holes.

Black Holes

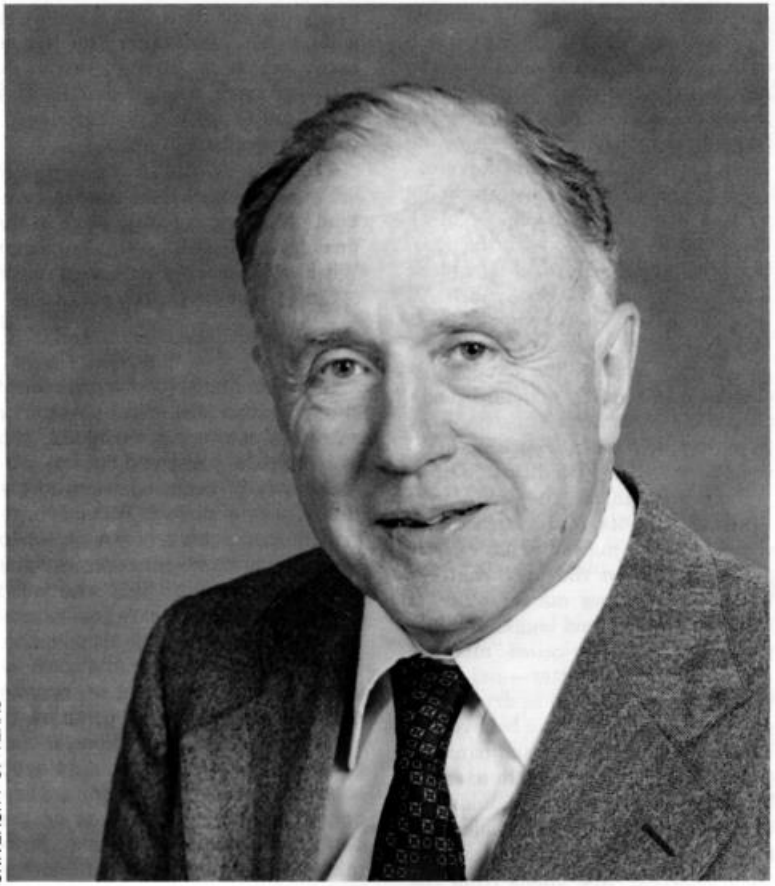
- Einstein published his paper on GR in Nov 1915.
- Karl Schwarzschild (1873-1916), German physicist turned soldier, found black holes mathematically in 1916.

- Schwarzschild died on the Russian front in May 1916 from disease.

We did not kill him!!!

Black Holes

- The term itself coined by John Wheeler (1911 – 2008) in 1967.
- There are many ways to think about black holes. The best one is a “river of space” analogy.

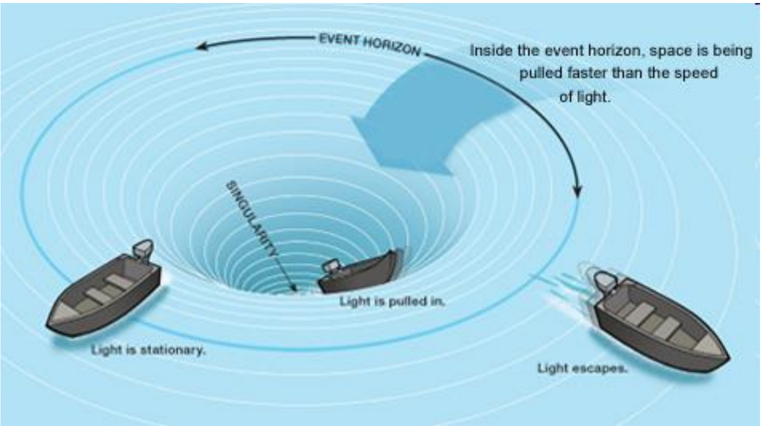


UNIVERSITY OF TEXAS

Wheeler

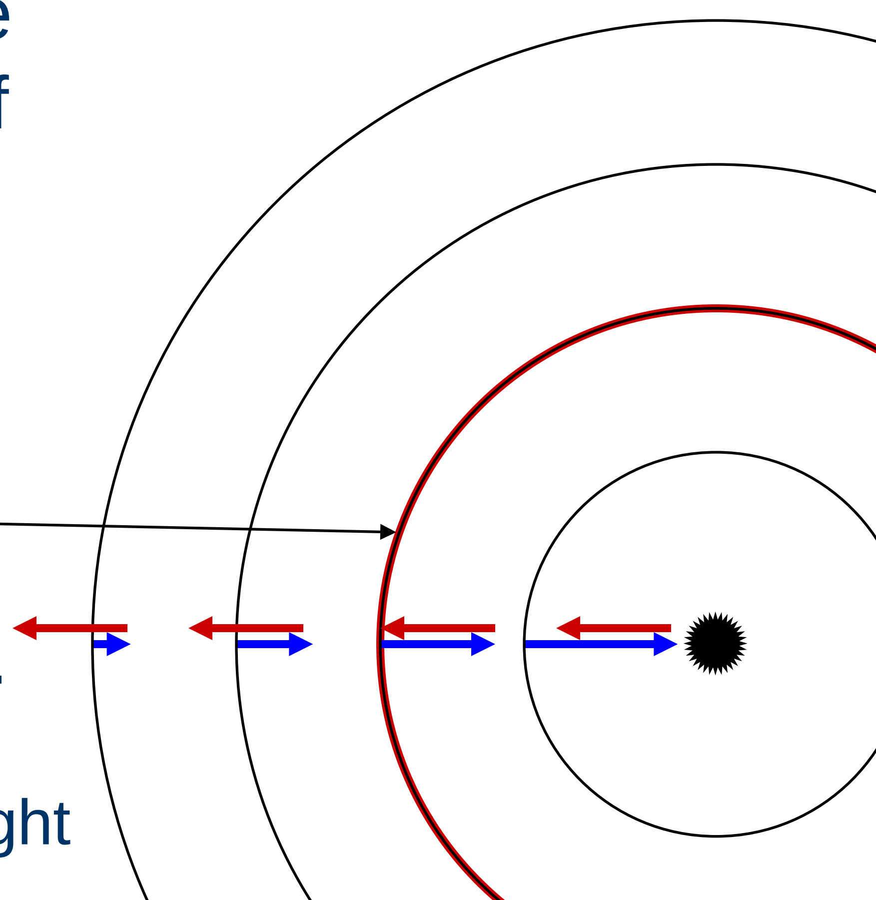
River Model of Black Holes

- Boat swimming against the river current can move around in slow places, but falls down the waterfall.



River Model of Black Holes

- In GR one cannot move faster than the speed of light ***relative*** to space.
- Radius where space flows in at ***c*** is called the ***horizon*** (a point of no return) or ***Schwarzschild radius***.



Speed of light

Flowing Space

- If the space flows into a black hole, wouldn’t it all disappear eventually? ***No*** – there are no laws for the “conservation of space”. Space can be destroyed and created.
- The universe expands because the space between galaxies expands – there is “more” space today than we had yesterday. That’s ok – the mess in my kid’s room does the same….

Schwarzschild radius

- The Schwarzschild radius (the “radius” of the horizon), is proportional to the mass of the black hole.
- If the Sun became a black hole (in reality it will not), its Schwarzschild radius will be just 3 km (1.8 miles).